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On Approximation of x^N by Incomplete Polynomials

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1. INTRODUCTION AND RESULTS

Let $\pi_{n,m}$ be the collection of all polynomials of degree n, leading coefficient equal to one, and divisible by x^{n-m}. Hence, each $p \in \pi_{n,m}$ is of the form $p(x) = x^n + d_1 x^{n-1} + \cdots + d_m x^{n-m}$. For each class $\pi_{n,m}$, let $p_{n,m}^*$ be the (unique) polynomial in $\pi_{n,m}$ with minimum supremum norm on $[0, 1]$; that is,

$$\| p_{n,m}^* \|_\infty = \inf\{\| p \|_\infty : p \in \pi_{n,m}\},$$

where $\| f \|_\infty = \sup\{| f(x)| : 0 \leqslant x \leqslant 1\}$. For example, $p_{n,n}^* = 2^{-n+1} T_n$, where $T_n(x) = \cos(n \cos^{-1} x)$ are the Chebyshev polynomials. It is easy to verify that not all the coefficients of $p_{n,n}^*$ are bounded as n tends to infinity. In fact, the "middle" coefficients of $p_{n,n}^*$ have the order of magnitude $n^{-1/2}$ $(27/16)^{n/4}$. In this paper, we show that if m does not tend to infinity with n, then all the coefficients of $p_{n,m}^*$ are bounded. This is included in the following

THEOREM 1. *Let m be a positive integer. Then there exist positive constants c_1 and c_2 such that*

$$c_1 n^{-m} \leqslant \| p_{n,m}^* \|_\infty \leqslant c_2 n^{-m} \tag{1.1}$$

for all $n > m$. Furthermore, the coefficients of $p_{n,m}^*$ are bounded as $n \to \infty$.

We remark that this theorem can be generalized. In fact, if the exponents $n - m, \ldots, n - 1$ in $p(x)$ above are replaced by integers $\lambda_1(n), \ldots, \lambda_m(n)$, respectively, where $0 \leqslant \lambda_1(n) < \cdots < \lambda_m(n) < n$, then the same conclusions of Theorem 1 still hold as long as $n - \lambda_1(n)$ is bounded as a function of n. This result which is contained in Theorem 2 is stated and proved in Section 4. Results analogous to Theorems 1 and 2 also hold for L^p, $1 \leqslant p \leqslant \infty$. These

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are stated as Theorem 3 in Section 5. This problem is inspired by the work of Lorentz and Zeller on approximation by incomplete polynomials (cf. [4, 5]). A related but somewhat different question was considered in [7, 8]. In [1], the authors answered a question of Lorentz and two of the results in [1] are used in the proof of Theorems 1 and 2.

Our approach to this problem is to compare it with the $L_2[0, 1]$ approximation problem where "everything" can be done explicitly. We therefore, devote the next section to the study of the $L_2[0, 1]$ problem.

Let $\Lambda_{k,N} = \{\boldsymbol{\lambda}_N = (\lambda_1, ..., \lambda_k)\}$, where $\lambda_j = \lambda_j(N), j = 1, ..., k$, are integers with $0 \leqslant \lambda_1 < \cdots < \lambda_k$, and for each $\boldsymbol{\lambda}_N \in \Lambda_{k,N}$ let $S(\boldsymbol{\lambda}_N)$, $\boldsymbol{\lambda}_N = (\lambda_1, ..., \lambda_k)$, be the vector space spanned by $\{x^{\lambda_1}, ..., x^{\lambda_k}\}$. Let e_{λ_N} be the $L_2[0, 1]$ error function obtained by approximating x^N from $S(\boldsymbol{\lambda}_N)$; that is,

$$e_{\lambda_N}(x) = x^N - \sum_{j=1}^{k} a_j x^{\lambda_j} \tag{1.2}$$

and

$$\| e_{\lambda_N} \|_2 = \inf\{\| x^N - p \|_2 : p \in S(\boldsymbol{\lambda}_N)\} \tag{1.3}$$

where $\| \cdot \|_2$ denotes the usual L_2 norm on [0, 1]. We show that if $\max\{| N - \lambda_1(N) |, | N - \lambda_k(N) |\}$ is bounded as $N \to \infty$, then the coefficients of e_{λ_N} remain bounded. One, therefore, expects that under the same hypothesis, all the k positive zeros of e_{λ_N} would cluster around the point 1. This and more can be proved in the special case when $\boldsymbol{\lambda}_N$ has components consisting of consecutive integers with N deleted. The fact that e_{λ_N} has precisely k positive zeros can be seen by using the Descartes rule of signs and the alternating property of best $L_2[0, 1]$ approximation.

Let $\boldsymbol{\lambda}_{l,N} = (N - k + l, ..., N - 1, N + 1, ..., N + l)$, $1 \leqslant l \leqslant k - 1$, $\boldsymbol{\lambda}_{0,N} = (N - k, ..., N - 1)$ and $\boldsymbol{\lambda}_{k,N} = (N + 1, ..., N + k)$. We have the following

PROPOSITION 1. *Let $e_{\lambda_{l,N}}$ be the $L_2[0, 1]$ error function e_{λ_N} as defined in* (1.2) *and* (1.3) *with $\boldsymbol{\lambda}_N = \boldsymbol{\lambda}_{l,N}$. Then for all $l = 0, ..., k$ and all k and N with $N \geqslant k + l$, all the positive zeros of $e_{\lambda_{l,N}}$ lie in the interval* $[1 - k^2/2N, 1)$.

All the afore mentioned L_2 results will be used to prove Theorem 1.

2. BEST APPROXIMATION BY INCOMPLETE POLYNOMIALS IN L_2

Let $\Lambda_{k,N} = \{\boldsymbol{\lambda}_N = (\lambda_1, ..., \lambda_k) : \lambda_j = \lambda_j(N), 0 \leqslant \lambda_1 < \cdots < \lambda_k\}$ and $e_{\lambda_N}(x) = x^N - \sum_{j=1}^{k} a_j x^{\lambda_j}$ be defined as in the previous section. In this section, we study some important properties of e_{λ_N}.

LEMMA 2.1. *For all N and $\boldsymbol{\lambda}_N \in \Lambda_{k,N}$*

$$\| e_{\boldsymbol{\lambda}_N} \|_2 = \frac{1}{(2N+1)^{1/2}} \prod_{j=1}^{k} \left| \frac{N-\lambda_j}{N+\lambda_j+1} \right|. \tag{2.1}$$

The above distance formula can be derived in a standard way (cf. [2; 9, p. 98]). We also have explicit expressions for the coefficients a_j.

LEMMA 2.2. *For $j = 1,..., k$,*

$$a_j = -\prod_{\substack{t=1\\ t\neq j}}^{k} \frac{N-\lambda_t}{\lambda_t-\lambda_j} \cdot \prod_{t=1}^{k} \frac{\lambda_t+\lambda_j+1}{N+\lambda_t+1}. \tag{2.2}$$

In particular, if $\max(|N-\lambda_1|, |N-\lambda_k|)$ *is bounded, then the coefficients a_j, $j = 1,..., k$, are bounded as $N \to \infty$. Furthermore*

$$e_{\boldsymbol{\lambda}_N}(1) = 1 - \sum_{j=1}^{k} a_j = \prod_{j=1}^{k} \frac{N-\lambda_j}{N+\lambda_j+1}. \tag{2.3}$$

To prove the above lemma, we note that $a_1,..., a_k$, and $y = e_{\boldsymbol{\lambda}_N}(1)$ satisfy the linear system:

$$a_1 + \cdots + a_k + y = 1,$$

$$\sum_{j=1}^{k} \frac{1}{\lambda_j+\lambda_\nu+1} a_j = \frac{1}{N+\lambda_\nu+1}, \qquad \nu = 1,..., k.$$

Apply Cramer's rule to solve for y. Simplifying the determinants by means of induction, we obtain (2.3) (see also [2; 6, p. 35]). Again, solve for each a_j. By using (2.3), one can simplify the expression for a_j to obtain (2.2).

It is interesting to note that

$$\| e_{\boldsymbol{\lambda}_N} \|_2 = | e_{\boldsymbol{\lambda}_N}(1)|/(2N+1)^{1/2} \leqslant \| e_{\boldsymbol{\lambda}_N} \|_\infty/(2N+1)^{1/2}.$$

Next, we study the location of the positive zeros of $e_{\boldsymbol{\lambda}_{l,N}}$ when $\boldsymbol{\lambda}_N = \boldsymbol{\lambda}_{l,N}$. Write

$$e_{\boldsymbol{\lambda}_{l,N}}(x) = x^N - \sum_{j=1}^{k} a_j^* x^{\lambda_j^*},$$

where $a_j^* = a_j^*(l)$ and $\boldsymbol{\lambda}_{l,N} = (\lambda_1^*,..., \lambda_k^*)$ is defined as in the above section. As mentioned above, each $e_{\boldsymbol{\lambda}_{l,N}}$ has precisely k positive zeros. By the alternating property of $e_{\boldsymbol{\lambda}_{l,N}}$, it is clear that these zeros are distinct and lie in the interval $(0, 1)$. Let $x_j = x_j(l, N)$, $j = 1,..., k$, $0 < x_1 < \cdots < x_k < 1$, be these zeros. Then if $l = 0$, $x_1 + \cdots + x_k = a_k^*$; if $l = 1$, $x_1 + \cdots + x_k = 1/a_k^*$; and if

$2 \leqslant l \leqslant k$, $x_1 + \cdots + x_k = -a_{k+1}^*/a_k^*$. By using (2.2), it is straightforward to verify that for each $l = 0,..., k$, and all k and N with $N \geqslant k + l$,

$$x_1 + \cdots + x_k \geqslant k(1 - k/2N).$$

Hence, $x_1 + k - 1 \geqslant k(1 - k/2N)$, or $1 - k^2/2N \leqslant x_1 < 1$. This completes the proof of Proposition 1.

We also remark that $1 - k^2/2N \leqslant x_1 \leqslant 1 - k/2N$.

3. Proof of the Main Result

In this section, we prove the main theorem of this paper, namely Theorem 1. Again, let $e_{\lambda_{l,N}}$ be the error function e_{λ_N} when $\boldsymbol{\lambda}_N = \boldsymbol{\lambda}_{l,N}$. Denote by $\| \cdot \|_1$ the usual L_1 norm on [0, 1]. We need several lemmas.

LEMMA 3.1. *Let k be a positive integer and $0 \leqslant l \leqslant k$. Then*

$$\| e_{\lambda_{l,N}} \|_1 \leqslant 3^{2k} \frac{1}{N^{1/2}} \| e_{\lambda_{l,N}} \|_2 \tag{3.1}$$

for all sufficiently large N.

Proof. We write

$$\| e_{\lambda_{l,N}} \|_1 = \int_0^{1-k^2/2N} | e_{\lambda_{l,N}}(x)| \, dx + \int_{1-k^2/2N}^1 | e_{\lambda_{l,N}}(x)| \, dx. \tag{3.2}$$

By Proposition 1, we have

$$\begin{aligned}
\int_0^{1-k^2/2N} | e_{\lambda_{l,N}}(x)| \, dx &\leqslant B_N \int_0^{1-k^2/2N} x^{N-k+l} \prod_{j=1}^{k} (x_j - x) \, dx \\
&< B_N \int_0^{1-k^2/2N} x^{N-k+l}(1 - x)^k \, dx \\
&< B_N \int_0^1 x^{N-k+l}(1 - x)^k \, dx \\
&= B_N \frac{k!}{(N - k + l + 1) \cdots (N + l + 1)},
\end{aligned} \tag{3.3}$$

where $B_N = \max(1, | a_k^* |)$. For the second integral, we use Schwarz's inequality to obtain

$$\int_{1-k^2/N}^1 | e_{\lambda_{l,N}}(x)| \, dx < \frac{k}{(2N)^{1/2}} \| e_{\lambda_{l,N}} \|_2 . \tag{3.4}$$

We can now use (2.1) and (2.2) to obtain an upper bound of the integral in (3.3) in terms of $\| e_{\boldsymbol{\lambda}_{l,N}} \|_2$ and combine this estimate with (3.4) to arrive at (3.1). This completes the proof of the lemma.

We next give a lower bound estimate of the $L_\infty[0, 1]$ distance from x^N to $S(\boldsymbol{\lambda}_{l,N})$. Denote this distance by

$$d_\infty(x^N, S(\boldsymbol{\lambda}_{l,N})) = \inf\{\| x^N - p \|_\infty : p \in S(\boldsymbol{\lambda}_{l,N})\},$$

$l = 0,..., k$. We have the following

LEMMA 3.2. *For $l = 0,..., k$ and all sufficiently large N,*

$$d_\infty(x^N, S(\boldsymbol{\lambda}_{l,N})) \geqslant \frac{N^{1/2}}{3^{2k}} \| e_{\boldsymbol{\lambda}_{l,N}} \|_2 . \tag{3.5}$$

Proof. Again, for convenience in notation, write $\boldsymbol{\lambda}_{l,N} = (\lambda_1^*,..., \lambda_k^*)$. Since $e_{\boldsymbol{\lambda}_{l,N}}$ is orthogonal to $x^{\lambda_1^*},..., x^{\lambda_k^*}$, we have

$$\| e_{\boldsymbol{\lambda}_{l,N}} \|_2^2 = (x^N, e_{\boldsymbol{\lambda}_{l,N}}) = \int_0^1 x^N e_{\boldsymbol{\lambda}_{l,N}}(x)\, dx.$$

Consider the measure

$$d\mu^*(x) = e_{\boldsymbol{\lambda}_{l,N}}(x)\, dx$$

and apply the duality theorem (cf. [9, p. 71]) to obtain

$$\begin{aligned} d_\infty(x^N, S(\boldsymbol{\lambda}_{l,N})) &= \sup \left\{ \left| \int_0^1 x^N \, d\mu(x) \right| \Big/ \int_0^1 | d\mu(x)| : d\mu \in S(\boldsymbol{\lambda}_{l,N})^\perp \right\} \\ &\geqslant \int_0^1 x^N \, d\mu^*(x) \Big/ \int_0^1 | d\mu^*(x)| \\ &= \| e_{\boldsymbol{\lambda}_{l,N}} \|_2^2 / \| e_{\boldsymbol{\lambda}_{l,N}} \|_1 . \end{aligned}$$

Hence, (3.5) follows from (3.1), and this completes the proof of the lemma.

The following result was obtained in [1].

LEMMA 3.3. *Let $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_k)$, $0 \leqslant \lambda_1 < \cdots < \lambda_k < N$. Then $d_\infty(x^N, S(\boldsymbol{\lambda}))$ is a decreasing function of each λ_j, $j = 1,..., k$.*

Hence, we have the following

LEMMA 3.4. *Let $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_k)$, $0 \leqslant \lambda_1 < \cdots < \lambda_k < N$ and $\bar{\boldsymbol{\lambda}} = (\lambda_1, \lambda_1 + 1,..., \lambda_1 + k - 1)$. Then*

$$d_\infty(x^N, S(\boldsymbol{\lambda})) \leqslant d_\infty(x^N, S(\bar{\boldsymbol{\lambda}})).$$

We are now ready to give an upper bound estimate of $d_\infty(x^N, \boldsymbol{\lambda}_N)$, where $\boldsymbol{\lambda}_N = (\lambda_1, ..., \lambda_k)$ satisfies $0 < \lambda_1 < \cdots < \lambda_k < N$.

LEMMA 3.5. *Let* $\boldsymbol{\lambda}_N = (\lambda_1, ..., \lambda_k)$ *where* $\lambda_j = \lambda_j(N)$, $1 \leqslant j \leqslant k$, *and* $0 \leqslant \lambda_1 < \cdots < \lambda_k < N$. *Suppose that*

$$N - \lambda_1(N) \leqslant A \tag{3.6}$$

for all large N. Then

$$d_\infty(x^N, S(\boldsymbol{\lambda}_N)) \leqslant CN^{-k}, \tag{3.7}$$

$C = 2^{A+1}k^k$, *for all sufficiently large N.*

Proof. Clearly, the function

$$x^{\lambda_1} \sum_{j=0}^{k-1} \binom{N - \lambda_1}{j} (x - 1)^j$$

is in $S(\bar{\boldsymbol{\lambda}}_N)$, where $\bar{\boldsymbol{\lambda}}_N = (\lambda_1, \lambda_1 + 1, ..., \lambda_1 + k - 1)$. Hence, by Lemma 3.4, we have

$$\begin{aligned}
d_\infty(x^N, S(\boldsymbol{\lambda}_N)) &\leqslant d_\infty(x^N, (S(\bar{\boldsymbol{\lambda}}_N)) \\
&\leqslant \left\| x^N - x^{\lambda_1} \sum_{j=0}^{k-1} \binom{N - \lambda_1}{j} (x - 1)^j \right\|_\infty \\
&= \left\| x^{\lambda_1}(1 - x)^k \sum_{j=k}^{N-\lambda_1} \binom{N - \lambda_1}{j} (x - 1)^{j-k} \right\|_\infty \\
&\leqslant \| x^{\lambda_1}(1 - x)^k \|_\infty \sum_{j=0}^{N-\lambda_1} \binom{N - \lambda_1}{j} \\
&\leqslant 2^A \| x^{\lambda_1}(1 - x)^k \|_\infty = 2^A \left(\frac{\lambda_1}{\lambda_1 + k}\right)^{\lambda_1} \left(\frac{k}{\lambda_1 + k}\right)^k \\
&\leqslant 2^{A+1}k^k N^{-k}
\end{aligned}$$

for all large N. This completes the proof of the lemma.

A less elementary and more precise upper bound estimate is given in [3, p. 125].

We are ready to prove Theorem 1. Let $p_{n,m}^* \in \pi_{n,m}$ be as defined in Section 1 and write

$$p_{n,m}^*(x) = x^n + c_1^* x^{n-1} + \cdots + c_m^* x^{n-m}.$$

Then $\| p^*_{n,m} \|_\infty = d_\infty(x^n, S(\bar{\lambda}))$, where $\bar{\lambda} = (n - m,..., n - 1)$. Hence, by applying Lemma 3.5, we have

$$\| p^*_{n,m} \|_\infty \leqslant c_2 n^{-m} \tag{3.8}$$

for all large n, where $c_2 = 2^{m+1}m^m$. To obtain a lower estimate, we use Lemma 3.2 with $l = 0$, $k = m$ and $N = n$, and apply formula (2.1). This gives

$$\| p^*_{n,m} \|_\infty \geqslant c_1 n^{-m}$$

for all large n with $c_1 = m!/2^{m+1}3^{2m}$. In order to prove the boundedness of the coefficients c_j^*, $j = 1,..., m$, we use the following trick pointed out to us by Professor P. Erdos. Let

$$| c_l^* | = \max\{| c_j^* |: 1 \leqslant j \leqslant m\}.$$

Then, using (3.8), we have

$$\begin{aligned} c_2 n^{-m} \geqslant \| p^*_{n,m} \|_\infty &= | c_l^* | \left| x^{n-l} - \sum_{j=1}^{m} b_j^* x^{\lambda_j} \right|_\infty \\ &\geqslant | c_l^* | \, d_\infty(x^{n-l}, S(\tilde{\lambda})) \end{aligned}$$

with appropriate definitions of $\bar{\lambda} = (\lambda_1 ,..., \lambda_m)$ and b_j^*'s. Hence, we can apply the lower bound estimate (3.5) in Lemma 3.2 and formula (2.1) in Lemma 2.1 to conclude that $| c_l^* | \leqslant Bn^m \cdot c_2 n^{-m} = c_2 B$ for some constant B and all large n. This completes the proof of Theorem 1.

4. A More General Result

In this section we prove that Theorem 1 remains valid under a more general setting. Let

$$\boldsymbol{\lambda}_N = (\lambda_1 ,..., \lambda_k),$$

where $\lambda_j = \lambda_j(N)$, $1 \leqslant j \leqslant k$, are integers with $0 \leqslant \lambda_1 < \cdots < \lambda_k < N$. Let $c_j^* = c_j^*(N), j = 1,..., k$, be the coefficient of the $L_\infty[0, 1]$ error function; that is,

$$p_N^*(x) = x^N - \sum_{j=1}^{k} c_j^* x^{\lambda_j}$$

and

$$\| p_N^* \|_\infty = d_\infty(x^N, S(\boldsymbol{\lambda}_N)).$$

We have the following result.

THEOREM 2. *Let* $\boldsymbol{\lambda}_N = (\lambda_1, ..., \lambda_k)$, $\lambda_j = \lambda_j(N)$, *be defined as above and suppose that*

$$N - \lambda_1(N) \leqslant D \tag{4.1}$$

for all N. *Then there exist positive constants* c_3 *and* c_4 *such that*

$$c_3 N^{-k} \leqslant \| p_N^* \|_\infty \leqslant c_4 N^{-k} \tag{4.2}$$

for all N. *Furthermore, the coefficients* $c_j^*(N), j = 1, ..., k$ *are bounded as a function of* N.

In order to prove this result, we need the following theorem established in [1].

THEOREM A. *Let* $\boldsymbol{\mu} = (\mu_1, ..., \mu_k)$, *where* $\mu_1, ..., \mu_k$ *are integers with* $0 \leqslant \mu_1 < \cdots < \mu_{k-l} < N < \mu_{k-l+1} < \cdots < \mu_k$ *and* $0 \leqslant l \leqslant k$. *Let* $\boldsymbol{\lambda}_{l,N}$ $0 \leqslant l \leqslant N$, *be as defined in Section* 1. *Then for each* l, $0 \leqslant l \leqslant k$ *and all* N,

$$d_\infty(x^N, S(\boldsymbol{\lambda}_{l,N})) \leqslant d_\infty(x^N, S(\boldsymbol{\mu})). \tag{4.3}$$

We now prove Theorem 2. The upper bound in (4.2) is precisely the result in Lemma 3.5. To get the lower bound, we simply apply Theorem A and Lemma 3.2 with $l = 0$, and then use Lemma 2.1. To prove that the coefficients $c_j^* = c_j^*(N)$ are bounded, we again let

$$| c_t^* | = \max\{| c_j^* |: 1 \leqslant j \leqslant k\}$$

and conclude that

$$c_4 N^{-k} \geqslant \| p_N^* \|_\infty = | c_t^* | \left\| x^{\lambda_t} - \sum_{j=1}^{k} d_j^* x^{\tilde{\lambda}_j} \right\|_\infty$$

$$\geqslant | c_t^* | \, d_\infty(x^{\lambda_t}, S(\tilde{\boldsymbol{\lambda}}))$$

with appropriate definitions of $\tilde{\boldsymbol{\lambda}} = (\tilde{\lambda}_1, ..., \tilde{\lambda}_k)$ and d_j^*'s. By Theorem A, with $l = k - t + 1$, we have

$$c_4 N^{-k} \geqslant | c_t^* | \, d_\infty(x^{\lambda_t}, S(\boldsymbol{\lambda}_{l,N})).$$

Hence, Lemma 3.2 applies and the same proof as that of Theorem 1 yields that $| c_t^* |$ is bounded. This completes the proof of the theorem.

5. Final Remarks

By a similar proof, we also obtain the following L^p result.

Theorem 3. *Let* $\boldsymbol{\lambda}_N = (\lambda_1(N),..., \lambda_k(N))$, $\lambda_1(N) < \cdots < \lambda_k(N) < N$ *and* $N - \lambda_1(N) \leqslant E$ *for all* N. *Let* p_N^{**} *be the error function obtained by approximating* x^N *from* $S(\boldsymbol{\lambda}_N)$ *in the* $L_p[0, 1]$ *norm*, $1 \leqslant p \leqslant \infty$. *Then there exist positive constants* c_5 *and* c_6 *such that*

$$c_5 N^{-k-(1/p)} \leqslant \| p_N^{**} \|_p \leqslant c_6 N^{-k-(1/p)}.$$

Furthermore, the coefficients of p_N^{**} *are bounded as a function of* N.

Theorem 1 leads to the following question: Let $p_{n,m_n}^* \in \pi_{n,m_n}$ satisfy $\| p_{n,m_n}^* \| = \inf\{\| p \|_\infty : p \in \pi_{n,m_n}\}$, where $m_n \to \infty$ *as* $n \to \infty$. To what extent does Theorem 1 hold and are the coefficients of p_{n,m_n}^* no longer bounded? By using Lemmas 2.1 and 2.2, this question is completely answered in the L_2 ca .

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References

1. I. Borosh, C. K. Chui, and P. W. Smith, Best uniform approximation from a collection of subspaces, *Math. Z.* **156** (1977), 13–18.
2. E. W. Cheney, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966.
3. R. P. Feinerman and D. J. Newman, "Polynomial Approximation," Williams & Wilkins Baltimore, 1974.
4. G. G. Lorentz, Approximation by incomplete polynomials (problems and results), *in* "Proc. Rational Approximation" (E. B. Saff and R. S. Varga, Eds.), Academic Press, New York, to appear.
5. G. G. Lorentz and K. L. Zeller, Best approximation by incomplete polynomials, to appear.
6. L. Mirsky, "An Introduction to Linear Algebra," Oxford Univ. Press (Clarendon), London, 1955.
7. D. J. Newman and T. J. Rivlin, Approximation of monomials by lower degree polynomials, *Aequationes Math.* **14** (1976), 451–455.
8. R. D. Riess and L. W. Johnson, Estimates for $E_n(x^{n+2m})$, *Aequationes Math.* **8** (1972), 258–262.
9. H. S. Shapiro, "Topics in Approximation Theory," Lecture Notes in Mathematics, Springer-Verlag, New York, 1971.